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Liouville equation in 1/8 BPS geometries

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Abstract: We investigate the $\frac{1}{8}$ BPS geometries with $\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{SO}(4)\times R$ symmetry in IIB supergravity which were classified by Gava et al, (hep-th/0611065). It is desirable to have a complete set of differential equations imposed on the controlling functions such that they are not only necessary but also sufficient to produce supergravity solutions with those symmetries. We work on this issue and find a new differential equation for the controlling functions. For a special case, we exhaust all the remaining constraints and show that they reduce to one Liouville equation. The solutions of this equation produce geometries which are locally equivalent to the near horizon geometries of intersecting D3-branes.

Keywords: D-branes, AdS-CFT Correspondence, String Duality, Gauge-gravity correspondence.

Contents

1. Introduction

In studying AdS/CFT correspondence, it is an interesting subject to examine the duality at the regions in which the state is highly excited to the extent that the backreactions in the gravity side are not negligible. In this sense, recent developments in the analysis of BPS geometries in supergravity theories are important as possible sources of information, and those results deserve to be studied in more details. In [1], by analyzing the BPS condition in IIB supergravity, a class of $\frac{1}{2}$ BPS geometries with SO(4)$\times$SO(4)$\times R$ symmetry were concisely written in terms of one function on a three-dimensional subspace and one differential equation imposed on that function was obtained so that the geometries were classified by the boundary conditions on a two-dimensional plane.

After this work, several works have been done to treat more general situations ([2−7]). Among them we concentrate on the result of [5], in which the case of SU(2)$\times$U(1)$\times$SO(4)$\times$ R symmetry was studied and as a result, a class of $\frac{1}{8}$ BPS geometries were written in terms of four functions and four differential equations for them have been found. One of the tasks left to be done is to exhaust the constraints for the controlling functions coming from the BPS condition and the equations of motion so that they form a framework to produce solutions of the supergravity with the above symmetries. Another is to find implications for the dual field theories which may arise as a result of these analyses on the gravity sides.

In this paper, to contribute in these directions, we report some new facts about the geometries considered in [5]. First we find that the differential equations obtained in [5] are not sufficient to exclude all the geometries which does not solve the supergravity equations of motion and present an additional differential equation which should be imposed on the four controlling functions. Second we find a restricted class of geometries in which the four functions and the five differential equations reduce to two functions and two differential equations. We pick up all the remaining constraints imposed by the BPS condition and the equations of motion for this class of geometries and find that one of the two controlling functions must be constant. The differential equation imposed on the remaining function becomes a Liouville equation having its cosmological constant as a free parameter and all the geometries which correspond to solutions of that equation are locally equivalent to the near horizon geometries of intersecting D3-brane systems. Thus one of the above mentioned tasks is completed in this restricted case. In this second part, the roles of the new differential equation are very crucial. We also argue on the T-duality transformation to D1-D5 system and possible future directions.

Apart from the discovery of the new differential equation for the general geometries in the first part, the restriction we consider in the second part eliminates perhaps most informative geometries, that is, asymptotically $AdS_5 \times S^5$ geometries. Nevertheless we consider that the appearance of geometries with another asymptotics should be considered as an important property because in some sense it relates two class of geometries with different asymptotics. If this relation is interpreted as a relation between the dual CFTs, we obtain a strong support for AdS/CFT correspondence in backreacted region.

This paper is organized as follows. In section 2 we review the analysis of [5] and explain how the new differential equation appears. In section 3 we take a limit which reduces the expressions for the geometries to simple forms, exhaust the constraints for them and exhibit the roles of the new differential equation. In section 4 we discuss the possibilities for applying our result.

2. 1/8 BPS geometries with $\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{SO}(4) \times R$

The purpose of this section is to examine the result of [5] and point out the existence of an additional constraint (2.46). We start with a review of the analysis in [5], as the derivation of (2.46) is related to its details.

Setup. In [1], a class of type IIB $\frac{1}{2}$ BPS geometries consisting of the metric and five-from flux with $\mathrm{SO}(4) \times \mathrm{SO}(4) \times R$ symmetry has been obtained through the procedure in which two S^3s were set in the starting ansatz and the Killing spinor equation was analyzed leading to the result that a timelike Killing vector was found and constraints for the other components of the geometry were picked up. In [5], that analysis was extended to $\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{SO}(4) \times R$ case. The basic idea is that we replace one of the S^3s in [1] with a squashed S^3 to break the $\mathrm{SU}(2)_R$ in the isometry group $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ of S^3. The ansatz for the $\mathrm{SU}(2)_L \times \mathrm{U}(1) \times \mathrm{SO}(4)$ symmetric metric and five-form flux is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \rho_1^2\left[\sigma_{\hat{1}}^2 + \sigma_{\hat{2}}^2\right] + \rho_3^2\left(\sigma_{\hat{3}} - A_\mu dx^\mu\right)^2 + \tilde{\rho}^2 d\tilde{\Omega}_3^2 \tag{2.1}$$

$$\begin{aligned} F_5 = &-\Big(G_{\bar{\mu}\bar{\nu}} e^{\bar{\mu}} \wedge e^{\bar{\nu}} \wedge e^{\bar{\hat{1}}} \wedge e^{\bar{\hat{2}}} \wedge e^{\bar{\hat{3}}} + *_4\tilde{V} \wedge e^{\bar{\hat{1}}} \wedge e^{\bar{\hat{2}}} + *_4\tilde{g} \wedge e^{\bar{\hat{3}}}\Big) \\ &+ \Big(\tilde{G}_{\bar{\mu}\bar{\nu}} e^{\bar{\mu}} \wedge e^{\bar{\nu}} + \tilde{V}_{\bar{\mu}} e^{\bar{\mu}} \wedge e^{\bar{\hat{3}}} + \tilde{g} e^{\bar{\hat{1}}} \wedge e^{\bar{\hat{2}}}\Big) \wedge \tilde{\rho}^3 d\tilde{\Omega}_3. \end{aligned} \tag{2.2}$$

Here μ, ν take values 0,1,2,3, and $g_{\mu\nu}, \rho_1, \rho_3, \tilde{\rho}, A_\mu, G_{\mu\nu}, \tilde{G}_{\mu\nu}, \tilde{V}, \tilde{g}$ depend only on the four-dimensional coordinate x^μ. $\sigma_{\hat{i}}$s are the left-invariant 1-forms used for building the metrics of squashed three-spheres. The explicit forms of them are

$$\sigma_{\hat{1}} = -\frac{1}{2}\left(\cos\hat{\psi}d\hat{\theta} + \sin\hat{\psi}\sin\hat{\theta}d\hat{\phi}\right)$$
$$\sigma_{\hat{2}} = -\frac{1}{2}\left(-\sin\hat{\psi}d\hat{\theta} + \cos\hat{\psi}\sin\hat{\theta}d\hat{\phi}\right)$$
$$\sigma_{\hat{3}} = -\frac{1}{2}\left(d\hat{\psi} + \cos\hat{\theta}d\hat{\phi}\right) \tag{2.3}$$

(see appendix A for notations related to the symmetry). $e^{\bar{\mu},\bar{\nu}}, e^{\bar{\hat{1}},\bar{\hat{2}},\bar{\hat{3}}}$ are the vierbein 1-forms with their indices in the respective tangent subspaces. We take $e^{\bar{\hat{1}},\bar{\hat{2}},\bar{\hat{3}}}$ to be of the forms associated with $\sigma_{\hat{1},\hat{2},\hat{3}}$:

$$e^{\bar{\hat{1}},\bar{\hat{2}}} = \rho_1\sigma_{\hat{1},\hat{2}}, \quad e^{\bar{\hat{3}}} = \rho_3\left(\sigma_{\hat{3}} - A\right).$$

$*_4$ is the Hodge dual in the four-dimensional subspaces described by the first term in the metric. $d\tilde{\Omega}_3^2$ is a metric of S^3 and $d\tilde{\Omega}_3$ is its volume form. Because we have set the coefficient of $\sigma_{\hat{1}}^2, \sigma_{\hat{2}}^2$ equal, the translation of $\hat{\psi}$ gives the extra U(1) symmetry. The five-form F_5 must satisfy two constraints. One is the self-duality relation, that is $F_5 = *F_5$, which in our ansatz reduces to

$$G_2 = *_4\tilde{G}_2. \tag{2.4}$$

The other is the Bianchi identity $dF_5 = 0$.

Supersymmetry requires the existence of a Killing spinor η the conditions for which are the Killing spinor equation

$$\nabla_M\eta + \frac{i}{480}F_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\Gamma_M\eta = 0 \tag{2.5}$$

and the chirality condition $\Gamma_{11}\eta = \eta$ where $\Gamma_{11} \equiv \Gamma^{\bar{0}}\cdots\Gamma^{\bar{9}}$. To analyze these conditions, we decompose the Dirac matrices in ten dimensions as follows.

$$\Gamma^{\bar{\mu}} = \gamma^{\bar{\mu}} \otimes 1 \otimes 1 \otimes 1 \otimes 1, \quad \Gamma^{\bar{\hat{a}}} = \gamma_5 \otimes \tau_{\hat{a}} \otimes 1 \otimes \hat{\tau}_1, \quad \Gamma^{\bar{\tilde{a}}} = \gamma_5 \otimes 1 \otimes \tau_{\tilde{a}} \otimes \hat{\tau}_2. \tag{2.6}$$

Here $\gamma^{\bar{\mu}}$s are the Dirac matrices in four dimensions, the chirality matrix in this subspace is defined as $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\tau_{\hat{a}}, \tau_{\tilde{a}}, \hat{\tau}_{1,2}$ are Pauli matrices. We consider Killing spinors of the correspondingly decomposed form

$$\eta = \epsilon \otimes \hat{\chi} \otimes \tilde{\chi}$$

where ϵ is a eight-component spinor on which the first and last components of each product in (2.6) act. This decomposition reduces the chirality condition for η to

$$\gamma_5\hat{\tau}_3\epsilon = \epsilon. \tag{2.7}$$

Moreover we restrict $\hat{\chi}$ to a constant eigenvector of $\tau_{\hat{3}}$ and $\tilde{\chi}$ to a Killing spinor on the S^3:

$$\tau_{\hat{3}}\hat{\chi} = s\hat{\chi}, \qquad s = \pm 1$$
$$\nabla'_{\bar{\tilde{a}}}\tilde{\chi} = \frac{i}{2}b\tau_{\tilde{a}}\tilde{\chi}, \qquad b = \pm 1$$

where $\nabla'_{\tilde{a}}$s are the covariant derivatives in the unit radius S^3. From this point we use $\mu, \nu, \ldots$ to denote tensors with their indices raised or lowered by the metric of four-dimensional subspace $g_{\mu\nu}$. The Killing spinor equation (2.5) is expressed as follows.

$$\left[\nabla'_\rho - \frac{1}{4}s\rho_3 F_{\rho\nu}\gamma^\nu\gamma_5\hat{\tau}_1 + isA_\rho - \left(\frac{1}{4}\tilde{G}_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{2}s\tilde{V}_\mu\gamma^\mu\gamma_5\hat{\tau}_1 + \frac{i}{2}s\tilde{g}\right)\gamma_5\hat{\tau}_2\gamma_\rho\right]\epsilon = 0 \quad (2.8)$$

$$\left[\frac{i}{2}\frac{\rho_3}{\rho_1}\gamma_5\hat{\tau}_1 + \frac{1}{2}\not{\partial}\rho_1 + \rho_1\left(\frac{1}{4}\tilde{G}_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{2}s\tilde{V}_\mu\gamma^\mu\gamma_5\hat{\tau}_1 - \frac{i}{2}s\tilde{g}\right)\gamma_5\hat{\tau}_2\right]\epsilon = 0 \quad (2.9)$$

$$\left[\frac{i}{2}\left(2-\frac{\rho_3^2}{\rho_1^2}\right)\gamma_5\hat{\tau}_1+\frac{1}{2}\not{\partial}\rho_3+\frac{1}{8}s\rho_3^2F_{\mu\nu}\gamma^{\mu\nu}\gamma_5\hat{\tau}_1+\rho_3\left(\frac{1}{4}\tilde{G}_{\mu\nu}\gamma^{\mu\nu}-\frac{1}{2}s\tilde{V}_\mu\gamma^\mu\gamma_5\hat{\tau}_1+\frac{i}{2}s\tilde{g}\right)\gamma_5\hat{\tau}_2\right] \times\epsilon = 0 \quad (2.10)$$

$$\left[\frac{i}{2}b\gamma_5\hat{\tau}_2 + \frac{1}{2}\not{\partial}\tilde{\rho} - \tilde{\rho}\left(\frac{1}{4}\tilde{G}_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{2}s\tilde{V}_\mu\gamma^\mu\gamma_5\hat{\tau}_1 + \frac{i}{2}s\tilde{g}\right)\gamma_5\hat{\tau}_2\right]\epsilon = 0 \quad (2.11)$$

where ∇'_μs are the covariant derivatives in the four-dimensional slice and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ (we will denote this two-form as F_2 in many other places in this paper).

Analysis of the conditions. To extract constraints for the metric and five-form flux from the conditions for supersymmetry, we use real spinor bilinears

$$K_\mu = \bar{\epsilon}\gamma_\mu\epsilon, \quad L_\mu = \bar{\epsilon}\gamma_5\gamma_\mu\epsilon, \quad Y_{\mu\nu} = \bar{\epsilon}\gamma_{\mu\nu}\hat{\tau}_1\epsilon, \quad f_1 = i\bar{\epsilon}\hat{\tau}_1\epsilon, \quad f_2 = i\bar{\epsilon}\hat{\tau}_2\epsilon \quad (2.12)$$

where $\bar{\epsilon} \equiv \epsilon^\dagger\gamma_{\bar{0}}$. Using Fierz rearrangements, we can show that

$$K^2 = -L^2 = -f_1^2 - f_2^2, \quad K \cdot L = 0. \quad (2.13)$$

From the reduced Killing spinor equations (2.8), (2.9), (2.10) and (2.11), we can deduce various constraints for the components in (2.1), (2.2). One of them is

$$L = -\frac{\rho_1 f_1}{\rho_3\tilde{\rho}}dy$$

where $y \equiv \rho_1\tilde{\rho}$. Thus, regarding y as a coordinate, we see that L_y is the only non-vanishing component of L. Another constraint is

$$\nabla'_\mu K_\nu = -G_{\mu\nu}f_2 + \tilde{G}_{\mu\nu}f_1 - \frac{\rho_3}{2}F_{\mu\nu}f_2 s + \epsilon_{\mu\nu\rho\sigma}K^\rho V^\sigma s - \tilde{g}Y_{\mu\nu}s. \quad (2.14)$$

From this we see that K^μ is a Killing vector and hence it is possible to introduce a coordinate t such that $K^\mu\partial_\mu = \partial_t$. Using the remaining two coordinate degrees of freedom, the metric of the four-dimensional subspace which respects (2.13) reduces to

$$ds_4^2 = -\frac{1}{h^2}(dt + V_i dx_i)^2 + h^2\frac{\rho_1^2}{\rho_3^2}\left(\delta_{\bar{i}\,\bar{j}}\;\tilde{e}_i^{\,\bar{i}}\tilde{e}_j^{\,\bar{j}}dx_i dx_j + dy^2\right)$$

where i, j take values $1, 2$, and $h^{-2} = f_1^2 + f_2^2$.

Further investigations of (2.5) show that $\rho_1, \rho_3, \tilde{\rho}$ are t-independent and that all the spinor bilinears defined in (2.12) and all the components of the five-form flux and F_2 can be written in terms of $\rho_1, \rho_3, \tilde{\rho}, K_\mu, A_t$ and the Levi-Civita symbol $\epsilon_{\mu\nu\rho\sigma}$. For later convenience we present here the results of $f_1, f_2, \tilde{V}$ and F_2.

$$
\begin{aligned}
f_1 =& \tilde{\rho}, \quad f_2 = \rho_3\left(c + sA_t\right), \quad \tilde{V} = \frac{s}{4}\frac{1}{\rho_3\tilde{\rho}^3}d\left(b\rho_1^2\tilde{\rho}^2 - \rho_3\tilde{\rho}^2 f_2\right), \\
F_{\mu\nu} =& -\frac{2s}{\rho_3(f_1^2+f_2^2)}\Bigg[\left(2-\frac{\rho_3^2}{\rho_1^2}\right)\frac{1}{\rho_3}\epsilon_{\mu\nu}{}^{\rho\sigma}K_\rho L_\sigma + \frac{b}{\tilde{\rho}}\left(K_\mu L_\nu - K_\nu L_\mu\right) \\
&+ f_1\epsilon_{\mu\nu}{}^{\rho\sigma}K_\rho\partial_\sigma \ln\left(\rho_3\tilde{\rho}\right) + f_2\left(K_\mu\partial_\nu\ln\left(\rho_3\tilde{\rho}\right) - K_\nu\partial_\mu\ln\left(\rho_3\tilde{\rho}\right)\right) \\
&+2sf_1\left(K_\mu\tilde{V}_\nu - K_\nu\tilde{V}_\mu\right) - 2sf_2\epsilon_{\mu\nu}{}^{\rho\sigma}K_\rho\tilde{V}_\sigma\Bigg] \qquad (2.15)
\end{aligned}
$$

where c is an integral constant of the differential equation for f_2. In solving the differential equation for f_1, noting that the sign of f_1 is flipped by the redefinition $\epsilon \to \hat{\tau}_3\epsilon$ without the chirality condition (2.7) affected, we have set f_1 positive, and in solving the differential equation for f_2, noting that $F_{\mu\nu}$ is t-independent, we have chosen a gauge in which A_μ is t-independent. We now set $A_y = 0$ by using the remaining gauge degrees of freedom.

Next we consider the constraints for the eight-component spinor ϵ. We have three projection conditions and hence one complex degrees of freedom is left for ϵ. The first projection is the chirality condition (2.7). The second comes from the relative normalization of $K_{\bar{0}}$ and $L_{\bar{3}}$,[1] and the third comes from the sum of (2.9) and (2.11) divided by $\rho_1, \tilde{\rho}$ respectively. To express these conditions in a simple way, we use a spinor $\epsilon_0 \equiv f_2^{-1/2}e^{-i\delta\gamma_5\gamma_3\hat{\tau}_1}\epsilon$ where δ is defined by $\sinh 2\delta = f_1/f_2$. The results are

$$
\gamma_5\hat{\tau}_3\epsilon_0 = \epsilon_0, \quad \gamma_{\bar{1}}\gamma_{\bar{2}}\epsilon_0 = -i\epsilon_0, \quad \gamma_{\bar{3}}\hat{\tau}_1\epsilon_0 = \epsilon_0 \qquad (2.16)
$$

and the normalization of ϵ_0 is given by $\epsilon_0^\dagger\epsilon_0 = 1$. Let us take an explicit representation of the Dirac matrices

$$
\gamma^{\bar{0}} = i\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \gamma^{\bar{1}} = \begin{pmatrix} \tau_1 & \\ & -\tau_1 \end{pmatrix}, \quad \gamma^{\bar{2}} = \begin{pmatrix} \tau_2 & \\ & -\tau_2 \end{pmatrix}, \quad \gamma^{\bar{3}} = \begin{pmatrix} \tau_3 & \\ & -\tau_3 \end{pmatrix} \qquad (2.17)
$$

where $\tau_{1,2,3}$ are Pauli matrices. Then the solution of (2.16) is

$$
\epsilon_0 \propto \begin{pmatrix} 0 \\ 1 \\ 0 \\ i \\ \hline 0 \\ -1 \\ 0 \\ i \end{pmatrix} \qquad (2.18)
$$

[1] Throughout this paper we take the vierbein of the four-dimensional subspace as its non-vanishing components are given by

$$
e_t{}^{\bar{0}} = \frac{1}{h}, \quad e_{x_i}{}^{\bar{0}} = \frac{V_i}{h}, \quad e_{x_i}{}^{\bar{j}} = h\frac{\rho_1}{\rho_3}\tilde{e}_i{}^{\bar{j}}, \quad e_y{}^{\bar{3}} = h\frac{\rho_1}{\rho_3}.
$$

where we have expressed the components of the spinor in a manner that the Dirac matrices (2.17) act on the four elements in each block and $\hat{\tau}_{1,2,3}$ act on the two blocks.

In addition to the bilinears defined in (2.12), we can define another type of bilinears by transposing the spinors. Note that

$$\frac{i}{\sqrt{2}}\left(\hat{\tau}_2+\hat{\tau}_3\right)\epsilon_0 \propto \frac{i-1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \hline 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \tag{2.19}$$

We can remove the phase factor of this expression by a phase shift or a local Lorentz rotation generated by $\gamma^{\bar{1}}\gamma^{\bar{2}}$. Calling this factor $e^{i\lambda}$, we obtain a spinor $\epsilon_0' \equiv e^{-i\lambda}\frac{i}{\sqrt{2}}\left(\hat{\tau}_2+\hat{\tau}_3\right)\epsilon_0$ with the following properties.

$$\epsilon_0'^t\epsilon_0' = 1, \quad \gamma_5\hat{\tau}_2\epsilon_0' = \epsilon_0', \quad \gamma_{\bar{1}}\gamma_{\bar{2}}\epsilon_0' = -i\epsilon_0', \quad \gamma_{\bar{3}}\hat{\tau}_1\epsilon_0' = -\epsilon_0'. \tag{2.20}$$

We now define non-vanishing spinor bilinears[2]

$$\omega_\mu = \epsilon'^t\gamma^{\bar{2}}\gamma_\mu\epsilon', \quad W^1_{\mu\nu} = \epsilon'^t\gamma^{\bar{2}}\gamma_{\mu\nu}\hat{\tau}_1\epsilon', \quad W^3_{\mu\nu} = \epsilon'^t\gamma^{\bar{2}}\gamma_{\mu\nu}\hat{\tau}_3\epsilon'$$

where $\epsilon' \equiv e^{-i\delta\gamma_5\gamma_3\hat{\tau}_1}f_2^{1/2}\epsilon_0' \;\left(= ie^{-i\lambda}\left(\hat{\sigma}_2+\hat{\sigma}_3\right)\epsilon/\sqrt{2}\right)$. From the Killing spinor equation (2.5), we obtain

$$\begin{aligned}\partial_\mu\omega_\nu - \partial_\nu\omega_\mu = {}& \frac{1}{\rho_3}\left(2-\frac{\rho_3^2}{\rho_1^2}\right)W^3_{\mu\nu} + \left(\frac{2b}{\tilde{\rho}} - \frac{\rho_3}{\tilde{\rho}\rho_1^2}f_2\right)W^1_{\mu\nu} \\ & + \frac{1}{\rho_3\tilde{\rho}}\left[\omega_\mu\partial_\nu\left(\rho_3\tilde{\rho}\right) - \omega_\nu\partial_\mu\left(\rho_3\tilde{\rho}\right)\right] - 2is\left(A_\mu\omega_\nu - A_\nu\omega_\mu\right).\end{aligned} \tag{2.21}$$

The (y, x^i) component of this relation implies that

$$\partial_y\left(i\tilde{e}_i{}^{\bar{1}} + \tilde{e}_i{}^{\bar{2}}\right) = D\left(i\tilde{e}_i{}^{\bar{1}} + \tilde{e}_i{}^{\bar{2}}\right)$$

where

$$D = h^2\left[2\frac{\rho_1\tilde{\rho}}{\rho_3^2} - 2\frac{\tilde{\rho}}{\rho_1} + f_2\left(2\frac{b\rho_1}{\rho_3\tilde{\rho}} - 2\frac{f_2}{\tilde{\rho}\rho_1}\right)\right].$$

[2]The rotation by Pauli matrices in (2.19) is important in defining ω_μs. Note that the chirality condition $\gamma_5\hat{\tau}_3\epsilon = \epsilon$ gives

$$\epsilon^t\gamma^{\bar{2}}\gamma_{\bar{\mu}}\epsilon = -\epsilon^t\gamma^{\bar{2}}\gamma_5\gamma_{\bar{\mu}}\hat{\tau}_3\epsilon.$$

In our representation of Dirac matrices (2.17), $\gamma^{\bar{2}}$ is antisymmetric and the others are symmetric and hence $\gamma^{\bar{2}}\gamma_5\gamma_{\bar{\mu}}$ is anti-symmetric. This implies that the above bilinears must vanish. In contrast, the rotation (2.19) changes the chirality condition to the second expression in (2.20) and for this reason we have non-vanishing components of ω.

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From this it turns out that, performing a y-independent coordinate transformation for x_1, x_2, we can set the metric of (x_1, x_2) space proportional to δ_{ij}. Therefore the metric of the four-dimensional subspace reduces to

$$ds_4^2 = -\frac{1}{h^2}(dt + V_i dx_i)^2 + h^2 \frac{\rho_1^2}{\rho_3^2}\left(T^2(x,y)\left(dx_1^2 + dx_2^2\right) + dy^2\right) \tag{2.22}$$

where T satisfies a differential equation

$$\partial_y \ln T = D. \tag{2.23}$$

The (x^1, x^2) component of (2.21) implies that

$$sA_i = (sA_t + c - b)\, V_i + \frac{1}{2}\epsilon_{ij}\partial_j \ln T \tag{2.24}$$

and the (t, x_i) components of (2.21) imply that $c = b$.

Assembling the above results, we can write the components of the metric and five-form flux in a concise way. To do that we introduce three functions m, n, p which are defined by

$$\rho_1^4 = \frac{mp + n^2}{m} y^4, \quad \rho_3^4 = \frac{p^2}{m(mp + n^2)}, \quad A_t = bs\frac{n-p}{p}. \tag{2.25}$$

We can see that all the components in (2.1), (2.2) are expressed in terms of m, n, p and T. The easiest to see is

$$D = 2y\left(n + m - \frac{1}{y^2}\right). \tag{2.26}$$

Eq. (2.14) determines the metric component V as follows.

$$dV = by *_3 \left[dn + \left(nD + 2ym(n-p) + \frac{2n}{y}\right) dy\right]. \tag{2.27}$$

Here $*_3$ is the Hodge dual in the three-dimensional subspaces spanned by x_1, x_2 and y, the metric for which is given by the expressions inside the bracket of the second term in (2.22). From the Bianchi identities for F_5, it turns out that the following two forms are closed and hence it is possible to define the potentials for them.

$$\begin{aligned} \rho_1^2\rho_3 G_2 &= d(B_t(dt + V) + \hat{B}) \\ \tilde{\rho}^3\tilde{G}_2 + \frac{1}{2}\tilde{g}\rho_1^2\tilde{\rho}^3 F_2 &= d(\tilde{B}_t(dt + V) + \hat{\tilde{B}}). \end{aligned}$$

As we mentioned above (2.15), the fluxes are expressed by other degrees of freedom. Using those expressions, we obtain

$$\begin{aligned} B_t &= b\frac{y^2}{4}\frac{n}{m}, & d\hat{B} &= \frac{y^3}{4} *_3 [dp + 4yn(p-n)dy] \\ \tilde{B}_t &= \frac{y^2}{4}\frac{n - \frac{1}{y^2}}{p}, & d\hat{\tilde{B}} &= b\frac{y^3}{4} *_3 [dm + 2mDdy]. \end{aligned} \tag{2.28}$$

Thus we have succeeded in writing all the components of the metric and five-form flux in terms of the four function m, n, p, T.

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Actually there are constraints other than the one that the geometry is expressed by m, n, p and T in the above way. One is (2.23). We can find constraints also from the integrability of the expressions for $dV, d\hat{B}$ and $d\hat{\hat{B}}$

$$
\begin{aligned}
ddV &= 0 && (2.29)\\
dd\hat{B} &= 0 && (2.30)\\
dd\hat{\hat{B}} &= 0. && (2.31)
\end{aligned}
$$

Explicit forms of these differential equations are given later in this section.

The analysis to this point is essentially included in [5]. Since we have four differential equations for four functions m, n, p, T, we see that the whole dependence of the metric and the flux on the coordinates is determined (at least locally) by the boundary conditions for these functions on a plane in the $x_1 x_2 y$ space, which is a generalization of the result in [1] where we had one function and one differential equation imposed on it.

New constraint. However, we point out here that an additional constraint must be imposed on m, n, p and T so that the allowed solutions are more restricted. Note that (2.24) determines A_i with respect to m, n, p and T, and recall that we have obtained (2.15) before and that equation determines the field strength $F_2 \equiv dA$ in terms of m, n, p and T. Explicitly, from (2.15) we obtain

$$F = -bs\left(dt+V\right)\wedge d\left(\frac{n}{p}\right) - \frac{s}{2} *_3 \left[\left(4m - \frac{(n^2+mp)(4n+8m)}{p}y^2\right)dy - \frac{2n}{p}ydn - 2ydm\right]. \quad (2.32)$$

This must coincide with the expressions for F_{yi}, F_{ij} obtained by differentiating (2.24), that is

$$
\begin{aligned}
F_{yi} &= \partial_y\left(A_t V_i\right) + \frac{s}{2}\epsilon_{ij}\partial_j\partial_y \ln T\\
&= bs\partial_y\left(\frac{n}{p}\right)V_i + s\frac{n-p}{p}y\epsilon_{ij}\partial_j n + \frac{s}{2}\epsilon_{ij}\partial_j\left[2y\left(n+m-\frac{1}{y^2}\right)\right]\\
F_{12} &= \partial_1\left(A_t V_2\right) - \partial_2\left(A_t V_1\right) + \frac{s}{2}\left(-\partial_1^2 - \partial_2^2\right)\ln T\\
&= bs\partial_1\left(\frac{n}{p}\right)V_2 - bs\partial_2\left(\frac{n}{p}\right)V_1 + s\frac{n-p}{p}yT^2\left(nD + 2ym(n-p) + \frac{2n}{y}\right) - \frac{s}{2}\left(\partial_1^2+\partial_2^2\right)\ln T
\end{aligned}
$$

where we have used (2.25), (2.26) and (2.27). The comparison for F_{yi} gives no new information. The comparison for F_{ij} gives a new constraint

$$\frac{1}{2}\left(\partial_1^2+\partial_2^2\right)\ln T = -T^2 y\partial_y n - T^2 y\partial_y m + 2T^2\left(m - 2m^2y^2 - 4mny^2 - n^2y^2 + mpy^2\right). \quad (2.33)$$

One might suspect that (2.33) can be derived from (2.23), (2.29), (2.30), (2.31) and is not a new constraint. In section 3, we will exclude this possibility by presenting a solution for (2.23), (2.29), (2.30), (2.31) which does not solve (2.33) (see below (3.2)).

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Summary. Here we summarize the result of this section. In the remainder of the paper, we set $b = s = 1$. The expressions for the metric and the five-form flux are

$$\begin{aligned} ds^2 &= -h^{-2}\left(dt + V_i dx_i\right) + h^2 \frac{\rho_1^2}{\rho_3^2}\left(T^2\left(dx_1^2 + dx_2^2\right) + dy^2\right) + \tilde{\rho}^2 d\tilde{\Omega}_3^2 \qquad (2.34)\\ &\quad +\rho_1^2\left(\hat{\sigma}_1^2 + \hat{\sigma}_2^2\right) + \rho_3^2\left(\hat{\sigma}_3 - A_t dt - A_i dx^i\right)^2 \\ F_5 &= -\left(G_{mn} e^m \wedge e^n \wedge e^{\hat{1}} \wedge e^{\hat{2}} \wedge e^{\hat{3}} + *_4 \tilde{V} \wedge e^{\hat{1}} \wedge e^{\hat{2}} + *_4 \tilde{g} \wedge e^{\hat{3}}\right) \qquad (2.35)\\ &\quad + \left(\tilde{G}_{mn} e^m \wedge e^n + \tilde{V}_m e^m \wedge e^{\hat{3}} + \tilde{g} e^{\hat{1}} \wedge e^{\hat{2}}\right) \wedge \tilde{\rho}^3 d\tilde{\Omega}_3. \end{aligned}$$

h^2 and the components of the five-form are expressed by $\rho_1, \rho_3, \tilde{\rho}, V, A$ (or its field strength $F_2 \equiv dA$),$B_t, \hat{B}, \tilde{B}_t$ and $\hat{\tilde{B}}$.

$$\begin{aligned} h^{-2} &= \tilde{\rho}^2 + \rho_3^2\left(1 + A_t\right)^2 \\ \tilde{g} &= \frac{1}{2\tilde{\rho}}\left(1 - \frac{\rho_3^2}{\rho_1^2}\left(1 + A_t\right)\right) \\ \tilde{V} &= \frac{1}{2\rho_3\tilde{\rho}^3} d\left(\tilde{g}\rho_1^2\tilde{\rho}^3\right) \\ \rho_1^2\rho_3 G &= d(B_t(dt + V) + \hat{B}) \\ \tilde{G}\tilde{\rho}^3 &= -\frac{1}{2}\tilde{g}\rho_1^2\tilde{\rho}^3 F_2 + d(\tilde{B}_t(dt + V) + \hat{\tilde{B}}). \end{aligned} \qquad (2.36)$$

The remaining degrees of freedom are further reduced to m, n, p and T by the following relations.

$$\rho_1^4 = \frac{mp + n^2}{m} y^4, \qquad \rho_3^4 = \frac{p^2}{m(mp + n^2)}$$

$$\tilde{\rho}^4 = \frac{m}{mp + n^2}, \qquad A_t = \frac{n - p}{p} \qquad (2.37)$$

$$dV = y *_3 \left[dn + \left(nD + 2ym(n - p) + \frac{2n}{y}\right) dy\right] \qquad (2.38)$$

$$A_i = A_t V_i + \frac{1}{2}\epsilon_{ij}\partial_j \ln T \qquad (2.39)$$

$$B_t = \frac{y^2}{4}\frac{n}{m}, \qquad d\hat{B} = \frac{y^3}{4} *_3 \left[dp + 4yn(p - n)dy\right] \qquad (2.40)$$

$$\tilde{B}_t = \frac{y^2}{4}\frac{n - \frac{1}{y^2}}{p}, \qquad d\hat{\tilde{B}} = \frac{y^3}{4} *_3 \left[dm + 2mDdy\right], \qquad (2.41)$$

where $D = 2y(m + n - 1/y^2)$. We have five differential equations for m, n, p, T

$$y^3\left(\partial_1^2 + \partial_2^2\right) n + \partial_y\left(y^3 T^2 \partial_y n\right) + y^2\partial_y\left[T^2\left(yDn + 2y^2 m(n - p)\right)\right] + 4y^2 D T^2 n = 0 \qquad (2.42)$$

$$y^3\left(\partial_1^2 + \partial_2^2\right) m + \partial_y\left(y^3 T^2 \partial_y m\right) + \partial_y\left(2y^3 T^2 m D\right) = 0 \qquad (2.43)$$

$$y^3\left(\partial_1^2 + \partial_2^2\right) p + \partial_y\left(y^3 T^2 \partial_y p\right) + \partial_y\left[4y^3 T^2 n y(n - p)\right] = 0 \qquad (2.44)$$

$$\partial_y \ln T = D. \qquad (2.45)$$

$$\frac{1}{2}\left(\partial_1^2 + \partial_2^2\right)\ln T = -T^2 y\partial_y n - T^2 y\partial_y m + 2T^2\left(m - 2m^2y^2 - 4mny^2 - n^2y^2 + mpy^2\right). \qquad (2.46)$$

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((2.42), (2.43) and (2.44) are the explicit forms of (2.29), (2.30) and (2.31) respectively.)

We have written down many constraints derived from the Bianchi identity, the self-duality relation and the Killing spinor equation, but it is uncertain whether we have equivalently transformed those original constraints. Moreover we need to impose the Einstein equation

$$R_{\mu\nu} = \frac{1}{6} F_{\mu\alpha\beta\gamma\delta} F^{\alpha\beta\gamma\delta}{}_{\nu} \tag{2.47}$$

on the above geometries. In the next section, we work on this issue for a restricted case of m and n, and show that the above results are insufficient to produce solutions of the supergravity with the symmetries required in the setup.

3. Deviation from LLM with $D = 0$, $\rho_1 = \rho_3$, and n fixed

The result in the previous section is a generalization of that in [1](LLM). A limit to LLM solutions is given by $\rho_3 = \rho_1, A_t = 0, T = const.$, in other words it is $D = 0, n = p, T = const.$. In this limit, all the degrees of freedom reduce to one function and the differential equation imposed on it can be solved by integral forms for general boundary conditions. However, in general case, although we have obtained differential equations (2.42), (2.43), (2.44), (2.45), (2.46) for the controlling functions m, n, p, T, it is far more difficult to solve them or find physical implications from them. Therefore we seek limits in which those equations reduce to tractable forms such that we can find meaningful information from them.

One of the chief interests would be on the property of our geometries near LLM ansatz. Paying attention to (2.37), we find that setting $\rho_1 = \rho_3$ almost gives another S^3 metric but this condition leaves two of the three degrees of freedom m, n, p. If we further set $D = 0$, $n - p$ is left as a deformation function for a special case of LLM. Expanding (2.37) in $n - p$ to the first order, we obtain

$$\rho_1^4 \sim \frac{ny^4}{1 - y^2 n} - y^4(n - p)$$
$$\rho_3^4 \sim \frac{ny^4}{1 - y^2 n} - \frac{1 + ny^2}{1 - ny^2} y^4(n - p).$$

Equating these two gives $n = 0, m = 1/y^2$. This condition is sufficient to satisfy $\rho_1 = \rho_3$ to all order and therefore we concentrate on these continuously deviated LLM geometries which have only two degrees of freedom p, T.[3]

In this case, the differential equations (2.42), (2.43), (2.44), (2.45), (2.46) are reduced to simple forms. Eq. (2.43) and (2.45) are equivalent and both imply that T is y-independent, $T = T(x)$. Then (2.42) implies that p is also y-independent, $p = p(x)$. and hence (2.44) reduces to a Laplace equation in two dimensions

$$\left(\partial_1^2 + \partial_2^2\right) p(x) = 0. \tag{3.1}$$

[3]For these geometries, n is fixed to 0 and p deviates from the LLM limit $n = p = 0$, but there is another solution for $\rho_1 = \rho_3, D = 0$, in which n also deviates from 0 and p, n satisfy a constraint $p - n = 2n/(y^2 n - 2)$. In this paper, we do not investigate this case and leave it for a future work.

Eq. (2.46) reduces to a simple but nonlinear equation

$$\left(\partial_1^2+\partial_2^2\right)\ln\left(T(x)^2\right)=8p(x)T(x)^2. \tag{3.2}$$

Now it is clear that (2.46) is independent from (2.42), (2.43), (2.44), (2.45). In our restricted case, the constraints of (2.42), (2.43), (2.44), (3.1) are equivalent to the requirement that p and T are y-independent and p satisfies (3.1), and hence they allow p and T to be constant, but that does not satisfy (3.2). Thus we can say that (2.46) is independent from (2.42), (2.43), (2.44), (2.45). Eq. (3.2) (in other words (2.46)) plays important roles in the remainder of this paper.

The other expressions in the result of the previous section also reduced to simple forms. We present some of them first. (2.38) becomes

$$dV=-2pT^2dx^1\wedge dx^2. \tag{3.3}$$

This equation for V can be solved by using (3.2). The solutions are given by

$$V_i=\frac{1}{4}\epsilon_{ij}\partial_j\ln T^2+\partial_i\alpha \tag{3.4}$$

where the first term is a particular solution guaranteed by (3.2) and α is an arbitrary function depending on x_1,x_2. The last expression in (2.37) reduces to $A_t=-1$, (2.39) reduces to $A_i=-\partial_i\alpha$, and hence $F_2=0$.

Straightforwardly we obtain reduced expressions for the metric and five-form flux

$$\begin{aligned}
ds^2 &= -\frac{1}{\sqrt{p}}(dt+V)^2+\sqrt{p}\left(T^2\left(x_1^2+x_2^2\right)+dy^2\right)+\frac{1}{\sqrt{p}}d\tilde{\Omega}_3^2\\
&\quad+\sqrt{p}y^2\left[\sigma_{\hat{1}}^2+\sigma_{\hat{2}}^2+\left(\sigma_{\hat{3}}+dt+\partial_i\alpha dx_i\right)^2\right] &(3.5)\\
F_5 &= -\rho_1^2\rho_3G_2\wedge\sigma_{\hat{1}}\wedge\sigma_{\hat{2}}\wedge\left(\sigma_{\hat{3}}+dt+\partial_i\alpha dx_i\right)\\
&\quad-\frac{p}{2}y^2*_4dy\wedge\sigma_{\hat{1}}\wedge\sigma_{\hat{2}}-\frac{\sqrt{p}}{2}y*_4 1\wedge\left(\sigma_{\hat{3}}+dt+\partial_i\alpha dx_i\right)\\
&\quad+\left(\tilde{\rho}^3\tilde{G}_2+\frac{y}{2}dy\wedge\left(\sigma_{\hat{3}}+dt+\partial_i\alpha dx_i\right)+\frac{y^2}{2}\sigma_{\hat{1}}\wedge\sigma_{\hat{2}}\right)\wedge d\tilde{\Omega}_3 &(3.6)
\end{aligned}$$

where

$$\begin{aligned}
\rho_1^2\rho_3G_2 &= \frac{y^3}{4}*_3dp &(3.7)\\
\tilde{\rho}^3\tilde{G}_2 &= \frac{1}{4p^2}dp\wedge(dt+V). &(3.8)
\end{aligned}$$

At this stage we can see that the self-duality relation (2.4) is restored by using the expressions (3.7), (3.8). Note that it is due to (3.2) that we deduced that $F_2=0$ and hence have the vanishing first term in (2.36).

Complete set of constraints. We have written down the reduced forms of the expressions in the summary of the previous section ((2.34)–(2.46)). We now exhaust all the other constraints for the above geometries.

First we reexamine the Killing spinor equation. In the previous section, the form of the spinor ϵ has been partly determined. Explicitly it is

$$\begin{aligned}\epsilon &= f_2^{\frac{1}{2}} e^{i\delta\gamma_5\gamma_3\hat{\tau}_1}\epsilon_0 \\ &= e^{i\left(\lambda-\frac{3}{4}\pi\right)} f_2^{\frac{1}{2}} e^{i\delta\gamma_5\gamma_3\hat{\tau}_1}\epsilon_c \\ &= e^{i\left(\lambda-\frac{3}{4}\pi\right)} f^{\frac{1}{2}} \left(\cosh\delta + i\sinh\delta\ \hat{\tau}_3\right)\epsilon_c \end{aligned} \tag{3.9}$$

where ϵ_c is the constant spinor in the right hand side of (2.18). In the third line we have used projection conditions in (2.16). From the expressions for f_1, f_2 in (2.15), we see that in our limit f_2 vanishes, hence e^δ diverges as $e^\delta \sim 2\left(\frac{f_1}{f_2}\right)^{1/2}$ and (3.9) converges to

$$\epsilon \sim e^{i\left(\lambda-\frac{3}{4}\pi\right)} f_1^{1/2}\left(1+i\hat{\tau}_3\right)\epsilon_c = p^{-1/8} e^{i\left(\lambda-\frac{3}{4}\pi\right)}\left(1+i\hat{\tau}_3\right)\epsilon_c.$$

We substitute this into (2.8), (2.9), (2.10), (2.11). Using (2.16) again, we obtain

$$\begin{aligned}\left(i\ \tilde{\not{N}}\hat{\tau}_3 - i\tilde{g}\gamma_5\hat{\tau}_2\right)\epsilon &= \frac{1}{2}p^{\frac{1}{4}}\left(i\gamma_3\hat{\tau}_3+\hat{\tau}_1\right)\epsilon \\ &= \frac{1}{2}p^{\frac{1}{8}} e^{i\left(\lambda-\frac{3}{4}\pi\right)}\left(i\gamma_3\hat{\tau}_3+\hat{\tau}_1\right)\left(1+i\hat{\tau}_3\right)\epsilon_c \\ &= 0.\end{aligned}$$

Thus we see that (2.9) and (2.10) are equivalent in our limit. Recall that the sum of (2.9) and (2.11) divided by $\rho_1, \tilde{\rho}$ is solved by the projection conditions for ϵ_0 (2.16). Therefore we consider only (2.8) and (2.9). To reexamine (2.8), we need the expression for the the spin connection $\omega_{\mu\bar{\nu}\bar{\rho}}$ in the four-dimensional subspace. Its non-vanishing components are shown to be

$$\begin{aligned}
&\omega_{t\bar{0}\bar{1}} = -\omega_{t\bar{1}\bar{0}} = \frac{\partial_1 p}{4p^{\frac{3}{2}}T}, && \omega_{t\bar{0}\bar{2}} = -\omega_{t\bar{2}\bar{0}} = \frac{\partial_2 p}{4p^{\frac{3}{2}}T}, \quad \omega_{t\bar{1}\bar{2}} = -\omega_{t\bar{2}\bar{1}} = -1 \\
&\omega_{x_1\bar{0}\bar{1}} = -\omega_{x_1\bar{1}\bar{0}} = \frac{\partial_1 p}{4p^{3/2}T}V_1, && \omega_{x_1\bar{0}\bar{2}} = -\omega_{x_1\bar{2}\bar{0}} = \frac{\partial_2 p}{4p^{3/2}T}V_1 - p^{1/2}T \\
&\omega_{x_2\bar{0}\bar{1}} = -\omega_{x_2\bar{1}\bar{0}} = \frac{\partial_1 p}{4p^{3/2}T}V_2 + p^{1/2}T, && \omega_{x_2\bar{0}\bar{2}} = -\omega_{x_2\bar{2}\bar{0}} = \frac{\partial_2 p}{4p^{3/2}T}V_2 \\
&\omega_{x_1\bar{1}\bar{2}} = -\omega_{x_1\bar{2}\bar{1}} = \frac{\partial_2 p}{4p} + V_1 - 2\partial_1\alpha, && \omega_{x_2\bar{1}\bar{2}} = -\omega_{x_2\bar{2}\bar{1}} = -\frac{\partial_1 p}{4p} + V_2 - 2\partial_2\alpha \\
&\omega_{y\bar{1}\bar{3}} = -\omega_{y\bar{3}\bar{1}} = -\frac{\partial_1 p}{4pT}, && \omega_{y\bar{2}\bar{3}} = -\omega_{y\bar{3}\bar{2}} = -\frac{\partial_2 p}{4pT}.
\end{aligned}$$

Using these expressions and the projection conditions (2.20), we obtain the reduced forms of (2.8)

$$\begin{aligned}\partial_t\left(p^{-1/8}e^{i\left(\lambda-\frac{3}{4}\pi\right)}\left(1+i\hat{\tau}_3\right)\epsilon_c\right) &= 0 \\ \left[\partial_{x_1} + \frac{1}{8}\left(\partial_{x_1}p\right)\right]\left(p^{-1/8}e^{i\left(\lambda-\frac{3}{4}\pi\right)}\left(1+i\hat{\tau}_3\right)\epsilon_c\right) &= 0\end{aligned}$$

$$\left[\partial_{x_2}+\frac{1}{8}\left(\partial_{x_2}p\right)\right]\left(p^{-1/8}e^{i\left(\lambda-\frac{3}{4}\pi\right)}\left(1+i\hat{\tau}_3\right)\epsilon_c\right)=0$$
$$\partial_y\left(p^{-1/8}e^{i\left(\lambda-\frac{3}{4}\pi\right)}\left(1+i\hat{\tau}_3\right)\epsilon_c\right)=0,$$

which leads to that $\lambda = const.$. We can show that (2.9) reduces to a trivial equation and gives no new constraint. Thus we see that the Killing spinor equation (2.5) only determines the phase factors of the Killing spinors and gives no new constraint for the metric and five-form flux (3.4), (3.5), (3.6), (3.7), (3.8).

Next we consider the Einstein equation (2.47). For convenience, we rewrite the expressions for the metric and five form flux in the following way. First we perform coordinate transformations $t \to t-\alpha, \hat{\psi} \to \hat{\psi} - t$ where $\hat{\psi}$ is a coordinate of the squashed three-sphere (see (2.3)). Note that the second transformation just absorbs the dt accompanied by $\sigma_{\hat{3}}$ and does not affect the other components of the metric and five-form flux:

$$\sigma_{\hat{3}}+dt\to\sigma_{\hat{3}},\quad \left(\sigma_{\hat{1}}\right)^2+\left(\sigma_{\hat{2}}\right)^2\to\left(\sigma_{\hat{1}}\right)^2+\left(\sigma_{\hat{2}}\right)^2,\quad \sigma_{\hat{1}}\wedge\sigma_{\hat{2}}\to\sigma_{\hat{1}}\wedge\sigma_{\hat{2}}.$$

We now see that another S^3 metric $d\hat{\Omega}^2 \equiv \sigma_{\hat{1}}^2+\sigma_{\hat{2}}^2+\sigma_{\hat{3}}^2$ appears in the metric (3.5). We parametrize that S^3 with a unit vector in four-dimensional space $\hat{\boldsymbol{y}} = (\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4)$, regard y as the coordinate of the radial direction and introduce coordinates $y_{1,2,3,4} \equiv y\hat{y}_{1,2,3,4}$. We have the relations

$$\begin{aligned}
dy^2+y^2\left(\sigma_{\hat{1}}^2+\sigma_{\hat{2}}^2+\sigma_{\hat{3}}^2\right) &= dy_1^2+dy_2^2+dy_3^2+dy_4^2\\
ydy\wedge\sigma_{\hat{3}}+y^2\sigma_{\hat{1}}\wedge\sigma_{\hat{2}} &= ydy\wedge\sigma_{\hat{3}}+\frac{y^2}{2}d\sigma_{\hat{3}}\\
&= -\frac{1}{2}R^1_{\alpha\beta}\left(\hat{y}_\alpha dy+yd\hat{y}_\alpha\right)\wedge\left(\hat{y}_\beta dy+yd\hat{y}_\beta\right)\\
&= -dy_1\wedge dy_2-dy_3\wedge dy_4
\end{aligned}$$

(see (A.3), (A.4)). Using these for (3.5), (3.6), we obtain the following expressions for the metric and five-form flux.

$$\begin{aligned}
ds^2 &= -\frac{1}{\sqrt{p}}\left(dt+V\right)^2+\sqrt{p}T^2\left(dx_1^2+dx_2^2\right)+\sqrt{p}\left(dy_1^2+dy_2^2+dy_3^2+dy_4^2\right)+\frac{1}{\sqrt{p}}d\tilde{\Omega}_2^2\\
V_i &= \frac{1}{4}\epsilon_{ij}\partial_j\ln T^2\\
F_5 &= \frac{1}{2}\left(\partial_2pdx_1-\partial_1pdx_2\right)\wedge dy_1\wedge dy_2\wedge dy_3\wedge dy_4+\frac{pT^2}{2}dt\wedge dx_1\wedge dx_2\wedge\left(dy_1\wedge dy_2+dy_3\wedge dy_4\right)\\
&\quad+\frac{1}{2p^2}\left(\partial_1pdx_1+\partial_2pdx_2\right)\wedge\left(dt+V\right)\wedge d\tilde{\Omega}-\frac{1}{2}\left(dy_1\wedge dy_2+dy_3\wedge dy_4\right)\wedge d\tilde{\Omega}.
\end{aligned}$$

The (t,t) component of (2.47) for this geometry is calculated to be

$$\begin{aligned}
0 &= R_{tt}-\frac{1}{6}F_{t\alpha\beta\gamma\delta}F^{\alpha\beta\gamma\delta}{}_t\\
&= -\frac{3}{4p^3T^2}\left(\left(\partial_1p(x)\right)^2+\left(\partial_2p(x)\right)^2\right),
\end{aligned}$$

which implies that $\partial_1 p = \partial_2 p = 0$, that is, p is constant.

We have shown that the metric and the five form flux are expressed with one constant parameter p in the following way.

$$ds^2 = -\frac{1}{\sqrt{p}}(dt+V)^2 + \sqrt{p}T^2\left(dx_1^2+dx_2^2\right) + \sqrt{p}\left(dy_1^2+dy_2^2+dy_3^2+dy_4^2\right) + \frac{1}{\sqrt{p}}d\tilde{\Omega}_3^2 \tag{3.10}$$

$$F_5 = \frac{p}{2}T^2 dt\wedge dx_1\wedge dx_2\wedge(dy_1\wedge dy_2+dy_3\wedge dy_4) - \frac{1}{2}(dy_1\wedge dy_2+dy_3\wedge dy_4)\wedge d\tilde{\Omega}_3. \tag{3.11}$$

$$V = \frac{1}{4}\epsilon_{ij}\partial_j \ln T^2 dx_i. \tag{3.12}$$

Because p is constant, the remaining known constraint (3.2) is a Liouville equation with a cosmological constant $-16p$

$$\left(\partial_1^2+\partial_2^2\right)\ln\left(T(x)^2\right) = 8pT(x)^2. \tag{3.13}$$

As we will see below, the solutions of this equation correspond to geometries which are locally equivalent to the near horizon geometry of intersecting D3-brane systems. This implies that all of them are solutions of the supergravity and hence no additional constraint arises from the other components of the Einstein equation (2.47).

$\boldsymbol{AdS_3\times S^3\times R^4}$**.** The general solution of (3.13) has been known through the study of two dimensional surface. On each connected domain in x_1x_2 space, it is of the form

$$T^2 du d\bar{u} = \frac{1}{p}\frac{\partial\xi(u)\bar{\partial}\bar{\xi}(\bar{u})}{\left|\xi(u)-\bar{\xi}(\bar{u})\right|^2}du d\bar{u} \tag{3.14}$$

where $u\equiv x^1+ix^2$ and ξ is an arbitrary holomorphic function. From the point of view of the global structure of the surface, u is the coordinate of a local patch inside the upper half plane or its quotient by the discrete subgroup Γ of the Möbius group SL$(2,R)$ and ξ is the local coordinate of the surface with which the metric is expressed in the standard form $ds_2^2\propto d\xi d\bar{\xi}/(\mathrm{Im}\xi)^2$. The solutions are classified by the matrices $M\in\Gamma$ which act on $\xi(u)$ as u goes around fixed points of Γ: 1)$|\mathrm{Tr}M|<2$ (elliptic), 2)$|\mathrm{Tr}M|=2$ (parabolic), 3)$|\mathrm{Tr}M|>2$ (hyperbolic). In this paper, we do not investigate the global structures of the solutions, and in that case, it is sufficient to consider one solution of (3.13) because the other solutions are related to it by coordinate transformation at least locally.

Let us consider an example of parabolic solution $T^2 = 1/4px_1^2$. Then, from (3.10), (3.11), (3.12), we obtain the following metric and the five-form.

$$ds^2 = -\frac{1}{\sqrt{p}}\left(dt+\frac{1}{2x_1}dx_2\right)^2 + \frac{1}{4\sqrt{p}x_1^2}\left(dx_1^2+dx_2^2\right) + \sqrt{p}\left(dy_1^2+dy_2^2+dy_3^2+dy_4^2\right) + \frac{1}{\sqrt{p}}d\tilde{\Omega}_3^2 \tag{3.15}$$

$$F_5 = \frac{1}{8x_1^2}dt\wedge dx_1\wedge dx_2\wedge(dy_1\wedge dy_2+dy_3\wedge dy_4) - \frac{1}{2}(dy_1\wedge dy_2+dy_3\wedge dy_4)\wedge d\tilde{\Omega}_3. \tag{3.16}$$

The last two terms of the metric represent R^4 and S^3 respectively. We can show that the three-dimensional space spanned by (t,x_1,x_2) is AdS_3. One way to do that is to show that its metric satisfies three-dimensional Einstein equation with a negative cosmological

	w_0	w_1	y_1	y_2	y_3	y_4	z_1	z_2	z_3	z_4
D3	○	○	○	○						
D3	○	○			○	○				

Table 1: Configuration of the branes.

constant. This is sufficient to study local issues because the local structures of three-dimensional gravity are governed by its cosmological constant. Another way is to present explicit coordinate transformations which lead to standard expressions for AdS_3, which will be more useful for the studies of global issues in the future. One such transformation is given by

$$\begin{aligned} x_1 &= \frac{z^2}{1+x^{+2}} \\ x_2 &= x^- - z^2 \frac{x^+}{1+x^{+2}} \\ t &= \arctan x^+ \end{aligned} \tag{3.17}$$

and this leads to a Poincaré metric of AdS_3 with radius $1/p^{\frac{1}{4}}$.[4] Explicitly (3.15), (3.16) become

$$ds^2 = \frac{1}{\sqrt{p}} \frac{-dx^+ dx^- + dz^2}{z^2} + \sqrt{p}\left(dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2\right) + \frac{1}{\sqrt{p}} d\tilde{\Omega}_3^2 \tag{3.18}$$

$$F_5 = \frac{1}{4z^3} dx^+ \wedge dx^- \wedge dz \wedge (dy_1 \wedge dy_2 + dy_3 \wedge dy_4) - \frac{1}{2}(dy_1 \wedge dy_2 + dy_3 \wedge dy_4) \wedge d\tilde{\Omega}_3. \tag{3.19}$$

The above ten-dimensional space is the near horizon geometry of an intersecting D3-brane system. To see this, let us consider a stack of D3-branes such that all the branes extend in 1+1 directions w_0, w_1 and localize in four directions z_1, z_2, z_3, z_4 (overall transverse space) and the remaining world volume directions are y_1, y_2 or y_3, y_4 (relative transverse space). This configuration is summarized in table 1. The supergravity solution which in a sense corresponds to this configuration is given as follows (see, for a review [8]).

$$\begin{aligned} ds^2 &= H_1^{-\frac{1}{2}} H_2^{-\frac{1}{2}}\left(-dw_0^2 + dw_1^2\right) + H_1^{-\frac{1}{2}} H_2^{\frac{1}{2}}\left(dy_1^2 + dy_2^2\right) + H_1^{\frac{1}{2}} H_2^{-\frac{1}{2}}\left(dy_3^2 + dy_4^2\right) \\ &\quad + H_1^{\frac{1}{2}} H_2^{\frac{1}{2}} \sum_{i=1}^{4} dz_i^2 \end{aligned} \tag{3.20}$$

$$\begin{aligned} F_5 &= -\frac{1}{2} dw_0 \wedge dw_1 \wedge dr \wedge \left(\frac{l_1}{r^3} H_1^{-2} dy_1 \wedge dy_2 + \frac{l_2}{r^3} H_2^{-2} dy_3 \wedge dy_4\right) \\ &\quad - \frac{1}{2} d\Omega_3 \wedge (l_2 dy_1 \wedge dy_2 + l_1 dy_3 \wedge dy_4) \end{aligned} \tag{3.21}$$

$$H_1 = 1 + \frac{l_1}{r^2}, \quad H_2 = 1 + \frac{l_2}{r^2}. \tag{3.22}$$

[4] We can show that the AdS_3 written in the global coordinate is also covered by the coordinate system used in (3.15), (3.16).

Here $r \equiv \sqrt{z_1^2+z_2^2+z_3^2+z_4^2}$ is the radial coordinate in the overall transverse space, $d\Omega_3$ is the volume form of the unit radius three-sphere orthogonal to it in the same space and l_1, l_2 are constants proportional to $g_s^{1/2}\alpha'$. The near horizon geometry is given by the limit $\alpha' \to 0$ with $U = r/\alpha'$ fixed. After this limit is taken, (3.20), (3.21) becomes

$$\begin{aligned} ds^2 &= \alpha' \left[\sqrt{L_1 L_2} U^2 \left(-dw_0^2 + dw_1^2\right) + \sqrt{L_1 L_2}\frac{dU^2}{U^2} + \sqrt{L_1 L_2} d\Omega_3^2 \right] \\ &\quad + \frac{1}{\alpha' \sqrt{L_1 L_2}} \left(dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2\right), \\ F_5 &= -\frac{1}{2} U dw_0 \wedge dw_1 \wedge dU \wedge (dy_1 \wedge dy_2 + dy_3 \wedge dy_4) - \frac{1}{2} d\Omega_3 \wedge (dy_1 \wedge dy_2 + dy_3 \wedge dy_4). \end{aligned}$$

where $L_{1,2} \equiv l_{1,2}/\alpha'$ and we have redefined $U \to \sqrt{L_1 L_2} U, y_{1,2} \to y_{1,2}/\sqrt{\alpha' L_2},\ y_{3,4} \to y_{3,4}/\sqrt{\alpha' L_1}$. These expressions coincide with (3.18), (3.19) under the identifications $z = 1/U,\ p = \alpha'^{-2}(L_1 L_2)^{-1}$. This near horizon geometry has 16 supersymmetries and thus it can be seen that, in the case of the solution (3.15), (3.16), we have 12 enhanced supersymmetries in addition to the 4 supersymmetries obtained in the previous section. In the case of other solutions for the Liouville equation, those enhanced symmetries may be inconsistent with the global identifications in the upper half plane $\mathrm{Im}\xi > 0$, and therefore we expect that the geometries produced by generic solutions are less supersymmetric than the above geometry produced by a solution covering the whole of the upper half plane.

Since the geometries described by (3.10), (3.11), (3.12), (3.13) have turned out to be equivalent to the near horizon geometries of intersecting D3-brane systems, it is valuable to present here the expressions for some T-dual geometries in our coordinate system. First we write down the expression for a gauge potential A_4 of the five form flux F_5 in (3.11). Noting (3.12), (3.13) (or (3.3)), we obtain a solution of the equation $F_5 = dA_4$

$$A_4 = \frac{1}{4} dt \wedge V \wedge (dy_1 \wedge dy_2 + dy_3 \wedge dy_4) - \frac{1}{2}(dy_1 \wedge dy_2 + dy_3 \wedge dy_4) \wedge O \qquad (3.23)$$

where O is a two-form potential of $d\tilde{\Omega}_3$.

To take T-duals of (3.10), (3.23), we need the value of the dilaton ϕ_{IIB}. We set it equal to 0 because in that case we do not have to care about the difference between Einstein frame and string frame for the above geometries. Compactifying R^4 to T^4 and taking T-dual in the direction y_1, we obtain the following type IIA geometries in string frame.

$$ds^2 = -\frac{1}{\sqrt{p}}(dt+V)^2 + \sqrt{p}T^2\left(dx_1^2+dx_2^2\right) + \frac{1}{\sqrt{p}}dy_1^2 + \sqrt{p}\left(dy_2^2+dy_3^2+dy_4^2\right) + \frac{1}{\sqrt{p}}d\tilde{\Omega}_3^2 \qquad (3.24)$$

$$\begin{aligned} A_3 &= dt \wedge V \wedge dy_2 - 2 dy_2 \wedge O, \qquad \phi_{\mathrm{IIA}} = -\frac{1}{4}\ln p \qquad (3.25) \\ V &= \frac{1}{4}\epsilon_{ij}\partial_j \ln T^2 dx^i, \qquad \left(\partial_1^2+\partial_2^2\right)\ln\left(T(x)^2\right) = 8pT(x)^2. \end{aligned}$$

These are locally equivalent to the near horizon geometries of D2-D4 systems. Further, taking T-dual in the direction y_2, we obtain the following type IIB geometries in string frame.

$$ds^2 = -\frac{1}{\sqrt{p}}(dt+V)^2 + \sqrt{p}T^2\left(dx_1^2+dx_2^2\right) + \frac{1}{\sqrt{p}}\left(dy_1^2+dy_2^2\right) + \sqrt{p}\left(dy_3^2+dy_4^2\right) + \frac{1}{\sqrt{p}}d\tilde{\Omega}_3^2 \qquad (3.26)$$

$$A_2 = dt \wedge V - 2O, \qquad \phi_{\rm IIB} = -\frac{1}{2}\ln p \qquad (3.27)$$
$$V = \frac{1}{4}\epsilon_{ij}\partial_j \ln T^2 dx^i, \qquad \left(\partial_1^2 + \partial_2^2\right)\ln\left(T(x)^2\right) = 8pT(x)^2.$$

These are locally equivalent to the near horizon geometries of frequently-discussed D1-D5 systems.

Wick rotation. Finally, since we have understood the basic properties of the geometries with the $S^3 \times S^3$ factor, we comment on the possibility that there is a connection to the results of other works. Liouville theory has appeared also in other contexts as in [9, 10]. Our result is different from theirs in that (3.13) is a Euclidean Liouville equation. Interestingly, as we will see below, we can find analytic continuations to Minkowskian Liouville equations such that the resultant metrics and fluxes are again $AdS_3 \times S^3 \times R^4$ solutions of the same supergravity.

Let us recall the form of the general solution of the Liouville equation (3.14). We can always take $\xi = x'_1 + ix'_2$ as coordinates of (x_1, x_2) space, and because $\xi(u)$ is a holomorphic function, the conformal form of the metric in two-dimensional space $g_{ij} \sim T^2\delta_{ij}$ is not affected by this coordinate transformation. Thus we see that the metric and five-form flux expressed in the coordinate ξ are also in our ansatz and satisfy the same constraints. The difference from (3.15), (3.16) is just that x_1 and x_2 are interchanged and some signs are flipped. Explicitly, the metric and five-form flux are given by

$$ds^2 = -\frac{1}{\sqrt{p}}\left(dt + V\right)^2 + \frac{1}{4\sqrt{p}x'^2_2}\left(dx'^2_1 + dx'^2_2\right) + \sqrt{p}\left(dy^2 + y^2 d\hat{\Omega}_3^2\right) + \frac{1}{\sqrt{p}}d\tilde{\Omega}_3^2$$
$$F_5 = -\frac{1}{4}dt \wedge dV \wedge \left(dy_1 \wedge dy_2 + dy_3 \wedge dy_4\right) - \frac{1}{2}\left(dy_1 \wedge dy_2 + dy_3 \wedge dy_4\right) \wedge d\tilde{\Omega}_3$$
$$V = -\frac{1}{2x'_2}dx'_1.$$

From this we see that the Wick-rotations $x'_1 \to \pm ix'_1$ lead to that V turns into $\pm iV$, $T^2 = 1/4px'^2_2$ is unchanged and (3.13) turns into a Minkowskian Liouville equation. To keep the metric real, we need another Wick rotation, $t \to \pm it$. After this double Wick rotation, the five-form flux remains real, and hence these metric and the flux are again a solution of IIB supergravity. The coordinate transformation in the three dimensional subspace which leads to the Poincaré metric (In the case of (3.15), (3.16) it was (3.17).) can be used with the corresponding Wick rotations $x'^{+,-} \to \pm ix'^{+,-}$. Thus we see that this Wick rotated geometry is again an $AdS_3 \times S^3 \times R^4$ solution and it is the near horizon geometry of an intersecting D3 brane system.

The above Wick rotations work also for the cases of T-dualized geometries (3.24), (3.25), (3.26), (3.27). The appearance of Liouville theory for every slice of constant t may lead to some understanding of D1-D5 systems in the future.

4. Conclusion

In this paper, we have shown that a new differential equation should be imposed on the resultant controlling functions m, n, p, T of [5], and discussed the limit $n = 0, m = 1/y^2$, in which the new equation plays crucial roles to obtain some properties of the geometries.

Among the properties recognized in this paper, the appearance of Liouville theory in D1-D5 systems seems to be related most directly to other works. In [9, 10], it has been shown that some boundary dynamics in AdS_3 are related to Liouville theory. In [11], properties possessed by the solutions of Liouville equation have been found in supergravity. Investigating the precise relations of our result to those works will be our next task. The interesting point is that, in contrast to them, we have obtained Liouville equation itself in the bulk.

Next of interest is in the possibility that our results may relate the spectra of different conformal field theories. The world volume theories on intersecting D3-branes have been constructed in [12] and those of D1-D5 systems are often discussed. Although we have not investigated how the global structures of the two-dimensional surfaces described by the Liouville theory are related to those of the ten-dimensional geometries, we have shown that those structures are common to the T-dualized geometries. This implies that, if there exists a solution which is regarded as an excited state in the near horizon geometry of one configuration of D-branes, we can map it to those of T-dualized systems. This kind of duality relation in the gravity sides may lead to new understandings about the relations between the two different dual field theories.

Another of interest is in the relation between $AdS_3 \times S^3 \times R^4$ and $AdS_5 \times S^5$. In LLM, $AdS_5 \times S^5$ is described by a circular droplet with its radius equal to the square of that of the AdS_5, and the limit of large radius or small radius corresponds to the limit $n = p = 0, m = 1/y^2$ in our geometries, which can also be considered as a large radius limit of AdS_3. More generally, for any droplet configuration of LLM, this geometry can be obtained by looking closely around any point on the $y = 0$ plane. Although this is a singular geometry and the validity of supergravity approximation has to be discussed, it might imply something new about the behaviors of $\mathcal{N} = 4$ SYM in the corresponding limits.

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A. Building blocks of squashed three-spheres

In this section, we explain the relation between the unit vector in four-dimensional space used in section 3 and building blocks used for defining the metrics of squashed three-spheres.

Let us parametrize the unit vector in four-dimensional space as follows.

$$\begin{aligned}
\hat{y}_1 &= \cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2} \\
\hat{y}_2 &= -\cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2}
\end{aligned}$$

$$\hat{y}_3 = -\sin\frac{\theta}{2}\cos\frac{\psi-\phi}{2}$$
$$\hat{y}_4 = \sin\frac{\theta}{2}\sin\frac{\psi-\phi}{2}.$$

In this parametrization, the metric of S^3 is

$$\begin{aligned} ds^2 &= d\hat{y}_1^2 + d\hat{y}_2^2 + d\hat{y}_3^2 + d\hat{y}_4^2 \\ &= \frac{1}{4}\left(d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\phi d\psi\right). \end{aligned} \tag{A.1}$$

To define the metric of squashed three-spheres, we consider $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ generators,

$$L^1 = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}, \quad L^2 = \begin{pmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{pmatrix}, \quad L^3 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}, \tag{A.2}$$

$$R^1 = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, \quad R^2 = \begin{pmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}, \quad R^3 = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}. \tag{A.3}$$

We can construct left-invariant one-forms from $\mathrm{SU}(2)_R$ generators

$$\begin{aligned} R^1_{\alpha\beta}\hat{y}_\alpha d\hat{y}_\beta &= -\sigma_{\hat{3}} \\ R^2_{\alpha\beta}\hat{y}_\alpha d\hat{y}_\beta &= -\sigma_{\hat{1}} \\ R^3_{\alpha\beta}\hat{y}_\alpha d\hat{y}_\beta &= \sigma_{\hat{2}}. \end{aligned} \tag{A.4}$$

The metrics of $\mathrm{SU}(2)_L$ invariant squashed three-spheres are given by

$$ds^2 = r_{ij}\left(R^i_{\alpha\beta}\hat{y}_\alpha d\hat{y}_\beta\right)\left(R^j_{\gamma\delta}\hat{y}_\gamma d\hat{y}_\delta\right)$$

where r_{ij} is an arbitrary symmetric tensor. We can define the metrics of $\mathrm{SU}(2)_R$ invariant squashed three-spheres by using the $\mathrm{SU}(2)_L$ generators (A.2).

$$ds^2 = l_{ij}\left(L^i_{\alpha\beta}\hat{y}_\alpha d\hat{y}_\beta\right)\left(L^j_{\gamma\delta}\hat{y}_\gamma d\hat{y}_\delta\right).$$

If $r_{ij} = l_{ij} = \delta_{ij}$, the two metrics coincide and are equal to (A.1), and each symmetry is enhanced to $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$.

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